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☒ Additional inventors are being named on the ___ separately numbered sheets attached hereto

TITLE OF THE INVENTION (280 characters max)

Algorithm for Exact Computation of Coefficients of Universal Maximally Flat FIR Filters

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Algorithm for Exact Computation of Coefficients of Universal Maximally Flat FIR Filters

Saed Samadi and Akinori Nishihara

Abstract

The universal maximally flat FIR filters have an exact closed-from expression for the value of their impulse response coefficients. The expression involves multiple summations and binomial coefficients. The binomial coefficients may possibly have non-integer arguments. The direct computation of the impulse response coefficients of the filters for a given set of design parameters is computationally expensive and requires especial software routines for evaluation of large binomial coefficients. We show that the impulse response coefficients of the universal maximally flat FIR filters may be computed efficiently using a two-stage algorithm. For a filter of order N with K zeros at $z=-1$, the first stage of the algorithm uses a two-term recurrence to generate a sequence of length $N-K+1$. This sequence undergoes an N-step iterative process that involves additions, subtractions and divisions by 2. The algorithm does not involve any form of binomial cofficients. Moreover, the algorithm requires less number of arithmetical operations compared to a direct evaluation using the closed-form formula. The algorithm is mostly suitable for real-time generation of the coefficients on a DSP chip because it may be easily implemented in a computational environment that has restrictions on the available hardware or software resources.

Keywords

Digital filters, impulse response coefficients, FIR filters, maximally Flat filters, lowpass filters, fractional delay systems, halfband filters, Bernstein polynomials, forward shift operator, binomial coefficients.

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I. Introduction

More than thirty years have passed since the publication of the seminal paper of O. Herrmann on the design of maximally flat FIR digital filters [1]. Of the body of the related research that has been published since Herrmann's article [2]-[21] [1], the most significant contribution to the theory of lowpass maximally flat FIR filters has been Baher's results on the design of filters with simultaneous conditions on amplitude and group-delay response [8]. In recent years, it has been recognized that Baher's filters form a unifying class of maximally flat filters. This class encompasses various types of seemingly distinct systems [20]. Calling this class of filters the universal maximally flat digital filters, the authors simplified Baher's formula for the transfer function and showed that lowpass filters of even or odd lengths, with linear or nonlinear phase response characteristics, as well as fractional delay and half-band filters may be readily obtained by setting the design parameters in an appropriate manner. Other results show that Baher's universal maximally flat filters are exact and optimal with respect to polynomial signals [21]. Apart from their importance because of this remarkable property, some authors have envisaged other application scenarios for maximally flat filters. For instance, [10] discusses a situation where the use of an L_∞- or L_2-optimal filter may not result in a desirable design.

Notwithstanding universal maximally flat digital filters have an explicit formula for their transfer functions and, thus, may be designed readily using a computer algebra system, the simplified formulas provided in [20] are still unwieldy. A direct use of the formulas involves computation of binomial coefficients with possibly non-integer indices that appear as terms of the summand in a three-fold sum. Binomial coefficients with non-integer indices cannot be computed by the simple method of Pascal's triangle. Another common problem with computations involving the binomial coefficients is due to their tendency to introduce very large integers to the intermediate steps of the computation.

[1] We do not claim that the list of references given here is a comprehensive one. The cited papers are those with either theoretical or historical implications.

With the computer power at our disposal today, this may not be a serious impediment in many situations. However, we can still think of cases where limitations on hardware and software resources dictate the use of efficient means of computation of filter coefficients. It is desirable that such methods be free of binomial coefficients and have a simple iterative structure. Moreover, we know that if the design parameter associated with the group delay is limited to rational numbers, as is the case with many realistic settings, the values of the impulse response coefficients are rational numbers. In such cases, it is important to ensure that the answer to the numerical problem of computation of an impulse response coefficient is exactly a fraction, say 1/3, not a floating number that may get printed as "0.333333335". Therefore, it is desirable for a computation algorithm to maintain this property of the impulse response coefficients and yield rational values for rational design parameters.

The contribution of this paper in connection with the above computational issues is as follows. To avoid a direct use of the computationally expensive formula developed in [20] for the task of exact computation of the coefficients of universal maximally flat filters, we devise a recursive algorithm that is free from binomial coefficients of any form, requires less multiplications, and may be implemented using rational arithmetic. We start by deriving a simple recurrence relation for generation of a sequence that forms the coefficients of a Bernstein-form representation of the transfer function. The Bernstein-form representation of the transfer function was fully developed in [20]. Then, a simple method for converting the Bernstein-form polynomials to the power form is introduced. A variance of this method is derived to convert the representation of the transfer function given in [20] to the power-form representation using successive additions, subtractions and divisions by 2. An algorithm for computation of the impulse response coefficients is obtained by combining our recurrence relation and Bernstein-to-power conversion method.

The treatment is self-contained featuring complete proofs and supported by a diagram and an example. The mathematical results required for derivation of the algorithm are developed in Section II.

The detailed description of the algorithm together with a pseudo-code for its implementation are given in Section III. A concrete example is also worked out in this section. Conclusions are drawn in Section IV where the computational complexity of the algorithm is studied as well.

II. Mathematical Underpinning

It was shown in [20] that the transfer function of the universal maximally flat digital filters, a family of filters identical to those proposed by Baher [8], is given by

$$H_{N,K,d}(z) = \sum_{0 \le j \le N-K} b_j \left(\frac{1-z^{-1}}{2}\right)^j \left(\frac{1+z^{-1}}{2}\right)^{N-j}, \tag{1}$$

where the coefficients b_j are characterized by the three-term recurrence

$$j\, b_j + 2d\, b_{j-1} - (j - N - 2)\, b_{j-2} = 0, \quad j \ge 1, \tag{2}$$

with the initial values

$$b_0 = 1 \quad \text{and} \quad b_{-1} = 0. \tag{3}$$

By converting (1) to the power form representation

$$H_{N,K,d}(z) = \sum_{0 \le k \le N} h_k z^{-k} \tag{4}$$

the impulse response coefficients h_k become [20]

$$h_k = \sum_{0 \le j \le N-K} \sum_{0 \le p \le k} \sum_{0 \le i \le j} \frac{(-1)^{j-i+p} \binom{\frac{N}{2}-d}{i} \binom{\frac{N}{2}+d}{j-i} \binom{j}{p} \binom{N-j}{k-p}}{2^N}, \quad k = 0, \ldots, N. \tag{5}$$

Obviously, (5) is a computationally expensive formula. To ameliorate this situation, we devise an algorithm that exploits the combination of (1) and (2) to compute h_k for $k = 0, \ldots, N$.

First, let us write (1) in the traditional Bernstein form by introducing the sequence b'_j defined by

$$b'_j \stackrel{\text{def}}{=} \begin{cases} \dfrac{b_j}{\binom{N}{j}}, & 0 \le j \le N, \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

Then we have

$$H_{N,K,d}(z) = \sum_{0 \le j \le N-K} b'_j \binom{N}{j} \left(\frac{1-z^{-1}}{2}\right)^j \left(\frac{1+z^{-1}}{2}\right)^{N-j}. \tag{7}$$

This is the Bernstein form of the transfer function in the traditional sense. Although (7) has an additional binomial coefficient compared to (1), we will see later that the introduction of b'_j results in considerable simplification.

A recurrence for b'_j is obtained by substituting (6) into (2) and dividing both sides by $j\binom{N}{j} \neq 0$. Thus,

$$b'_j + 2d \frac{\binom{N}{j-1}}{j\binom{N}{j}} b'_{j-1} - (j-N-2) \frac{\binom{N}{j-2}}{j\binom{N}{j}} b'_{j-2} = 0, \quad 1 \le j \le N. \tag{8}$$

The above recurrence can be simplified by invoking the definition of the binomial coefficients [22] and a few simple algebraic steps. It follows that

$$b'_j = \frac{-2d}{N-j+1} b'_{j-1} - \frac{j-1}{N-j+1} b'_{j-2}, \quad 1 \le j \le N, \tag{9}$$

with the initial values

$$b'_0 = 1 \quad \text{and} \quad b'_{-1} = 0. \tag{10}$$

Thus, for given values of N and d we can compute the values of b'_j recursively by (9).

The next step is to convert (7) to the power form (4) and obtain the impulse response coefficients. A direct expansion of the summand in (7) using the binomial theorem will introduce additional sums and new binomial coefficients. Hence, this is not a desirable action to take. The other alternative is to use a conversion identity well-known in the mathematics literature and in the field of computer-aided geometric design. The identity is based on taking successive differences of the Bernstein coefficients. We state this identity as a proposition and provide a proof for it.

Proposition 1: Let the Bernstein form of the polynomial $p(x) = \sum_{0 \le i \le N} p_i x^i$ *be given by*

$$p(x) = \sum_{0 \le i \le N} b_i \binom{N}{i} x^i (1-x)^{N-i}. \tag{11}$$

Then

$$p_i = \binom{N}{i} \Delta^i b_0, \quad i = 0, \ldots, N, \tag{12}$$

where the operator Δ is the forward difference operator.

The forward difference operator is defined by the general recurrence

$$\Delta^j b_i = \Delta(\Delta^{j-1} b_i), \tag{13}$$

together with

$$\Delta^1 b_i = \Delta b_i = b_{i+1} - b_i, \quad \Delta^0 b_i = b_i. \tag{14}$$

For instance, $\Delta b_0 = b_1 - b_0$ and $\Delta^2 b_0 = b_2 - 2b_1 + b_0$. In this notation, the operator acts on the indices, written as subscripts, of the members of a sequence. To prove the proposition, we also need the forward shift operator E. The effect of E on the index i of a sequence like b_i is specified by

$$E^j b_i = E(E^{j-1} b_i). \tag{15}$$

together with

$$E^1 b_i = E b_i = b_{i+1}, \quad E^0 b_i = b_i. \tag{16}$$

E is related to Δ by

$$\Delta = E - 1. \tag{17}$$

Proof of proposition. We start with the Bernstein form of $p(x)$ and expand the term $(1-x)^{N-i}$ using the binomial theorem to obtain

$$p(x) = \sum_{0 \le i \le N} \sum_{0 \le j \le N-i} (-1)^{N-i-j} b_i \binom{N}{i} \binom{N-i}{j} x^{N-j}. \tag{18}$$

By the symmetry of the binomial coefficients and the trinomial revision property [22], we have

$$\binom{N}{i}\binom{N-i}{j} = \binom{N}{j}\binom{N-j}{N-i-j}. \tag{19}$$

On the other hand

$$b_i = E^i b_0. \tag{20}$$

Thus, we can write

$$p(x) = \sum_{0\le i\le N} \sum_{0\le j\le N-i} (-1)^{N-i-j} \binom{N}{j} \binom{N-j}{N-i-j} x^{N-j} E^i b_0. \tag{21}$$

The ranges for the indices of the two-fold sum above may be interchanged as

$$\sum_{0\le i\le N} \sum_{0\le j\le N-i} = \sum_{0\le j\le N} \sum_{0\le i\le j} \tag{22}$$

with no effect on the value of the sum. Now, we substitute $N-j$ for j in the sum and obtain

$$p(x) = \sum_{0\le N-j\le N} \sum_{0\le i\le j} (-1)^{N-i-(N-j)} \binom{N}{N-j} \binom{N-(N-j)}{N-i-(N-j)} x^{N-(N-j)} E^i b_0. \tag{23}$$

The right side above simplifies to

$$\sum_{0\le j\le N} \sum_{0\le i\le j} (-1)^{j-i} \binom{N}{j} \binom{j}{j-i} x^j E^i b_0. \tag{24}$$

Note that

$$\sum_{0\le i\le j} (-1)^{j-i} \binom{j}{j-i} x^j E^i b_0 = (E-1)^j b_0. \tag{25}$$

It then follows that

$$p(x) = \sum_{0\le j\le N} x^j \binom{N}{j} (E-1)^j b_0 \tag{26}$$

that is the power form representation of $p(x)$. From the uniqueness of the power from representation of a polynomial and the relation between the operators E and Δ it follows that $p_i = \binom{N}{i}\Delta^i b_0$. ∎

We now present our main and last result in this preparatory phase before proceeding to the derivation of the algorithm in the next section.

Corollary 1: The transfer function $H_{N,K,d}(z)$ *is given by*

$$H_{N,K,d}(z) = (\frac{1+E}{2} + \frac{1-E}{2} z^{-1})^N b_0'. \tag{27}$$

Proof. Let

$$z^{-1} = 1 - 2x. \tag{28}$$

Then, we can write

$$H_{N,K,d}(x) = \sum_{0 \le j \le N-K} b'_j \binom{N}{j} x^j (1-x)^{N-j}. \tag{29}$$

By the above proposition we have

$$H_{N,K,d}(x) = \sum_{0 \le j \le N} \binom{N}{j} \Delta^j b'_0 x^j. \tag{30}$$

By substituting for x in terms of z^{-1} in the above relation and expanding the summand using the binomial theorem, we obtain

$$H_{N,K,d}(z) = \sum_{0 \le j \le N} \binom{N}{j} \Delta^j b'_0 \sum_{i \le j} \binom{j}{i} 2^{-j} (-z^{-1})^i. \tag{31}$$

By the trinomial revision identity we have

$$\binom{N}{j}\binom{j}{i} = \binom{N}{i}\binom{N-i}{j-i}. \tag{32}$$

We can also modify the sum indices and write

$$H_{N,K,d}(z) = \sum_{0 \le i \le N} \binom{N}{i} \left(\frac{-\Delta z^{-1}}{2}\right)^i \sum_{i \le j} 2^{-(j-i)} \binom{N-i}{j-i} \Delta^{j-i} b'_0. \tag{33}$$

Using $p = j - i$ as an index for the second sum, we obtain

$$H_{N,K,d}(z) = \sum_{0 \le i \le N} \binom{N}{i} \left(\frac{-\Delta z^{-1}}{2}\right)^i \sum_{0 \le p \le N-i} \binom{N-i}{p} \left(\frac{\Delta}{2}\right)^p b'_0. \tag{34}$$

This simplifies to

$$H_{N,K,d}(z) = \sum_{0 \le i \le N} \binom{N}{i} \left(\frac{-\Delta z^{-1}}{2}\right)^i \left(1 + \frac{\Delta}{2}\right)^{N-i} b'_0. \tag{35}$$

By invoking the binomial theorem, we can further write

$$H_{N,K,d}(z) = \left(1 + \frac{\Delta}{2} - \frac{\Delta}{2} z^{-1}\right)^N b'_0. \tag{36}$$

By using the relation between Δ and E, (27) follows. ∎

III. The Algorithm

The algorithm consists of two major stages. In the first stage, we run the recurrence (9) from $j = 1$ up to $j = N - K$ to obtain the sequence

$$B' = (1, b_1' \ldots, b_{N-K}', 0, \ldots, 0). \tag{37}$$

The entries b_j' of B' for $N - K < j \leq N$ are set to zero. The next step is to expand the transfer function using (27). This can be done by noting that

$$\left(\frac{1+E}{2} + \frac{1-E}{2} z^{-1}\right)^p b_0' = \left(\frac{1+E}{2} + \frac{1-E}{2} z^{-1}\right) \left(\left(\frac{1+E}{2} + \frac{1-E}{2} z^{-1}\right)^{p-1} b_0'\right) \tag{38}$$

for arbitrary integer $p > 0$. Evaluation of both sides above in the power form yields

$$\sum_i h_i^{(p)} z^{-i} = \left(\frac{1+E}{2} + \frac{1-E}{2} z^{-1}\right) \sum_i h_i^{(p-1)} z^{-i}, \tag{39}$$

where $h_i^{(p)}$ is a sequence of coefficients defined by

$$h_i^{(p)} = (h_{i,j}^{(p)}) = (h_{i,0}^{(p)}, h_{i,1}^{(p)}, \ldots), \tag{40}$$

and operator E effects a forward shift on the index j. Thus, we obtain

$$\sum_i h_i^{(p)} z^{-i} = \sum_i \left(\frac{1+E}{2} h_i^{(p-1)} z^{-i} + \frac{1-E}{2} h_i^{(p-1)} z^{-i-1}\right). \tag{41}$$

It follows that

$$h_i^{(p)} = \frac{1+E}{2} h_i^{(p-1)} + \frac{1-E}{2} h_{i-1}^{(p-1)}. \tag{42}$$

The ranges for i and p are specified by

$$1 \leq p \leq N, \qquad 0 \leq i \leq p. \tag{43}$$

The initial values are given by

$$h_0^{(0)} = B', \quad h_{-1}^{(p)} = (0, 0, \ldots). \tag{44}$$

This completely specifies the recurrence we use in the second stage of our algorithm. After completing the Nth step, at the point when p has reached N, the value of the i-th impulse response coefficient is given by

$$h_i = \text{first term of sequence } h_i^{(N)}, \quad i = 0, \ldots, N. \tag{45}$$

The recurrence is illustrated in Fig. 1 for N = 3 and an unspecified value of K. It can be visually verified that the recurrence has a triangular structure. A pseudo-code of a procedure for implementation of the algorithm is given in Fig. 2.

definitions

h 1-D array for impulse response coefficients

B′ 1-D array for b'_j

r 3-D array for the intermediate values

i counter for r as depicted in Fig. 1

j counter for r and **B′**

p counter for r as depicted in Fig. 1

```
procedure GetCoefficients(N, K, d)
  for j = 0 to N step +1 do
    B′[j] ← 0 // Initialization
  endfor
  for i, j, p = 0 to N step 1 do
    r(p, i, j) ← 0 // Initialization
  endfor
  B′[1] ← 1 // Initial values
  for j = 1 to N − K step +1 do //      (A)
    B′[j] ← (−1/(N−j+1)) ((2d) B′[j − 1] + (j − 1) B′[j − 2])
  endfor //      END OF (A)
  for j = 0 to N − K step 1 do
    r[0, 0, j] ← B′[j] // Initial values
  endfor
  for p = 1 to N step 1 do //      (B)
```

```
for i from 0 to p step 1 do
  for j from 0 to N − p step 1 do
    r[p,i,j] := (r[p − 1,i − 1,j] − r[p − 1,i − 1,j + 1])/2 + (r[p − 1,i,j] + r[p − 1,i,j + 1])/2
  endfor
 endfor
endfor //    END OF (B)
for i = 0 to N step 1 do
  h[i] ← r[N,i,0] end do
return h
endprocedure
```

Fig. 2. Pseudo-code of a procedure that returns the values of impulse response coefficients as an array.

The following example helps elucidate the details. Let $N = 3$, $K = 1$ and $d = (-1)/4$. The actual values of the sequences $h_i^{(p)}$, $i = 0, \ldots, 3$, corresponding to the 3-D array r in the procedure of Fig. 2, are as follows

$$\begin{array}{lllll} & h_0^{(p)} & h_1^{(p)} & h_2^{(p)} & h_3^{(p)} \\ p=0: & (1, 1/6, -11/24, 0) & & & \\ p=1: & (7/12, -7/48, -11/48) & (5/12, 5/16, -11/48) & & \\ p=2: & (7/32, -3/16) & (35/48, 1/12) & (5/96, 13/48) & \\ p=3: & (1/64) & (39/64) & (31/64) & (-7/64) \end{array} \tag{46}$$

Only non-zero values of the sequences $h_i^{(p)}$, $i = 1, \ldots, 3$, are presented. Note that all computations have been done in the field of rational numbers in an exact manner. Hence, round-off errors are avoided.

IV. Conclusion and Remarks

We have shown that an alternative representation for the transfer function of universal maximally flat FIR filters enables us to develop a very simple algorithm for evaluation of the impulse response coefficients of the filters. The alternative form of the transfer function is based on the shift operator E and is derived by means of a technique referred to as finite calculus in [22].

The algorithm consists of two stages. In the first stage we run a two-term recurrence relation and obtain a 1-D sequence of numbers. If the value of the group delay parameter d is a rational number, then the outputs of the recurrence are rational. The outputs are then used in a simple N-step algorithm of triangular structure. In each step of this algorithm, the values of the sequences obtained in the preceding step undergo simple additions, subtractions and divisions by 2 to yield sequences that are shorter in size. Upon the completion of the Nth step, N sequences are obtained. The first entries of these sequences are the values of the impulse response coefficients. As long as the computations in the second stage are done in the field of rational numbers the final values remain rational. This provides

an accurate and fast means for exact computation of the impulse response coefficients.

A worst-case analysis of the computational complexity of the proposed algorithm is the final topic of this section. The computational complexity may be analyzed by evaluating the complexity of the simple for-loop marked by (A) and the nested for-loop designated by (B) in Fig. 2. For the number of additions and subtractions $C_{\pm}$, we obtain

$$C_{\pm} = \sum_{1 \le j \le N-K} 4 + \sum_{1 \le p \le N} \sum_{0 \le i \le p} \sum_{0 \le j \le N-p} 3. \tag{47}$$

For a closed-form expression, note that

$$C_{\pm} = 4(N-K) + 3 \sum_{1 \le p \le N} \sum_{0 \le i < p+1} (N-p+1), \tag{48}$$

that can be evaluated as

$$C_{\pm} = 4(N-K) + 3 \sum_{0 \le p < N} (p+2)(N-p). \tag{49}$$

Thus, we get

$$C_{\pm} = 4(N-K) + 3\left(\frac{5}{6}N + N^2 + \frac{1}{6}N^3\right). \tag{50}$$

For the number of multiplications $C_{\star}$, we have

$$C_{\star} = \sum_{1 \le j \le N-K} 2 = 2(N-K). \tag{51}$$

The number of divisions $C_{/}$ is given by

$$C_{/} = \sum_{1 \le j \le N-K} 1 + \sum_{1 \le p \le N} \sum_{0 \le i \le p} \sum_{0 \le j \le N-p} 2. \tag{52}$$

that becomes

$$C_{/} = (N-K) + 2\left(\frac{5}{6}N + N^2 + \frac{1}{6}N^3\right). \tag{53}$$

In comparison, for the direct evaluation using (5), the number of additions and subtractions $C_{\pm}^{\mathrm{Direct}}$ is

$$C_{\pm}^{\mathrm{Direct}} > \sum_{0 \le j \le N-K} \sum_{0 \le p \le k} \sum_{0 \le i \le j} 1 \tag{54}$$

The right side above is of the order $\mathcal{O}(N^4)$. The number of multiplications C_*^{Direct}, on the other hand, is

$$C_*^{Direct} > \sum_{0\leq j\leq N-K}\sum_{0\leq p\leq k}\sum_{0\leq i\leq j} j\,k. \tag{55}$$

The right side above is of the order $\mathcal{O}(N^5)$. For the number of divisions in a direct evaluation, we can write

$$C_/^{Direct} > \sum_{0\leq j\leq N-K}\sum_{0\leq p\leq k}\sum_{0\leq i\leq j} 4. \tag{56}$$

The right side above, again, is of order $\mathcal{O}(N^4)$. Thus, in addition to the simplicity of most of the division operations in the proposed algorithm (that are divisions by 2), a significant reduction of the overall complexity is accommodated in terms of sheer numbers.

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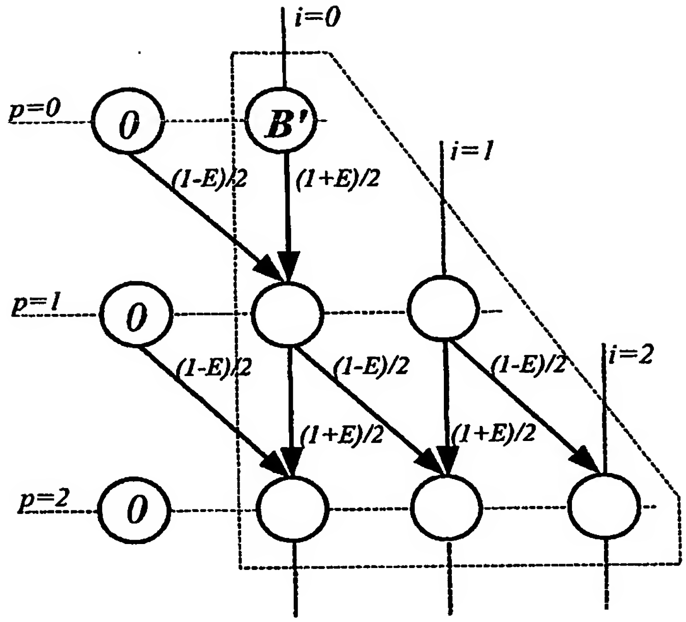

Fig. 1. Illustrative representation of the algorithm. Each node represents a sequence $h_i^{(p)}$. The nodes marked by 0's represent zero sequences.